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An Eulerian Code for the Study of the Drift-Kinetic Vlasov Equation

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An Eulerian code is developed to study the drift-kinetic Vlasov equation in a strong magnetic field. This is of interest in relation to studies on the $\mathbf{E} \times \mathbf{B}$ instabilities, as, for instance, the hydrodynamical Kelvin–Helmholtz (KH) instabilities and the ion temperature gradient instability. We investigate the nonlinear behavior of nonneutral plasma involving the formation and growth of vortices as a result of a velocity shear introduced by a nonuniform $\mathbf{E} \times \mathbf{B}$ drift. In particular the linear growth rates of KH instabilities have been calculated numerically and are shown to be in very good agreement with analytical theory. Preliminary results obtained for the ion temperature gradient instability are also presented. © 1993 Academic Press, Inc.

1. INTRODUCTION

Strong magnetic fields are commonly used in laboratory plasma experiments, especially for the important problem of plasma confinement in large devices. In many of these devices an electric field $\mathbf{E}$ exists which results in an $\mathbf{E} \times \mathbf{B}$ flow. Any inhomogeneity in these electric fields results in a shear in the plasma flow and can lead to the appearance of low-frequency instabilities. The physics of a plasma streaming across a static strong magnetic field has been the subject of intense theoretical, experimental, and computer simulation research. The basic equation to describe such a system is the drift-kinetic or guiding center Vlasov equation which couples the $\mathbf{E} \times \mathbf{B}$ motion across a magnetic field to the motion parallel to the magnetic field. This system possesses some interesting features, which makes it of interest as a dynamical system in its own right, in addition to its relevance for real plasmas in a strong magnetic field, as, for instance, the problem of instabilities associated with the $\mathbf{E} \times \mathbf{B}$ drift at the edge of a tokamak [1] and to the mathematically similar problem of the two-dimensional turbulence in an ideal fluid [2]. The nonlinear evolution of the instabilities can lead to the formation of vortices or convective cells (see [2–5]) which may have important consequences on the problem of plasma confinement and in particular in the problem of anomalous diffusion across a magnetic field. The recent interest in the problem of the physics of the edge of a tokamak is an important example [6–7].

Analytical and numerical results have been previously obtained for the study of the stability of the solutions of the linearized 2D vorticity equations and also for 3D fluid guiding-center plasmas [8–10]. In particular the growth rates of KH instabilities have been calculated analytically and numerically in [10, 11] for the case of a flow having a point of inflexion. Similar analysis has been developed in [12–14] to study the linear stability in inhomogeneous 3D guiding-center plasma with a *water-bag* model, where the drift-kinetic Vlasov description of a 3D guiding-center plasma has been investigated. It was shown that this model includes resonant three waves interaction between plasma waves at different angles to a strong external magnetic field, the interaction being of decay instability type [8, 13, 14]. Finally, we note the recent interest in ion-temperature-gradient instabilities, also known as η_i modes, in connection with $\mathbf{E} \times \mathbf{B}$ rotation and anomalous transport at the edge of tokamaks. In this case the adiabatic response of electrons to the perturbing potential precludes the existence of KH (driven by velocity shear) and Rayleigh–Taylor (driven, e.g., by a centrifugal acceleration) instabilities. For a comprehensive review and study of the simulation of the ion-temperature-gradient instabilities, see the recent work in Ref. [15]. An important problem in the study of these instabilities is the small growth rate of these low frequency instabilities and the relatively low saturation level of their

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potential, which makes their simulation using particle codes difficult. An obvious way to improve the understanding of these instabilities is to use an eulerian Vlasov model, where (unlike particles in cell codes) we have a good description of vortex structures and phase space resolution and very low numerical noise. Indeed clear pictures and results have been obtained by Vlasov codes in the precise mechanism of the electrostatic nonlinear evolution of phase space holes associated with strong nonlinear plasma oscillations (Bernstein–Greene–Kruskal equilibria, [16, 17]). Recently the eulerian Vlasov code has been applied to the study of the mechanism of accelerated particles to high energies with a very good resolution in the low density region to study the phase space dynamics of electrons accelerated by Raman scattering [18, 19]. These models remain one-dimensional in space. The purpose of the present work is to present a 2D drift-kinetic Vlasov code in order to describe the nonlinear behavior of plasma instabilities in an $\mathbf{E}\times\mathbf{B}$ flow and the resulting 2D vortex structure formation. In Section 2 we give the basic equations of the drift kinetic Vlasov model and then present the numerical algorithm. We compute in Section 3, in the case of a 2D guiding center approximation (for a perpendicular magnetic field) the growth rates of the KH instability and compare the results with the analytical values obtained in Ref. [11] using the Rayleigh stability equation. Nonlinear behavior and vortex formation in the $x-y$ plane and their mutual interaction (coalescence of vortices) are also investigated numerically. The simulation results for a tilted magnetic filed in the $x-z$ plane, using the drift-kinetic model are presented in Section 4. Finally in Section 5, we present preliminary results for a simulation to study the growth and saturation of an ion temperature gradient instability, and Section 6 will present our conclusion.

2. THE DRIFT-KINETIC VLASOV MODEL

2.1. *The Basic Equations and the Numerical Code*

In this section we present the basic equations that primarily govern the plasma dynamics in the case of a uniform magnetic field $B=(B_x, O, B_z)$ in the $x-z$ plane. We assume that the electrostatic shear instability takes place in a frequency regime lower than the ion cyclotron frequency and we use the guiding-center drift approximation. The electron and ion velocity can be written in the following form:

$$\mathbf{v}_{e,i}=\mathbf{v}_{\|e,i}+\mathbf{v}_{\perp} \qquad \text{with} \quad \mathbf{v}_{\perp}=\frac{\mathbf{E}\times\mathbf{B}}{B^2}.$$

The drift-kinetic Vlasov equation is written

$$\frac{\partial f_{i,e}}{\partial t}+\mathbf{v}_{\|}\cdot\frac{\partial f_{i,e}}{\partial \mathbf{r}_{\|}}+\frac{\mathbf{E}\times\mathbf{B}}{B^2}\cdot\frac{\partial f_{i,e}}{\partial \mathbf{r}_{\perp}}+\frac{e}{m_{i,e}}\mathbf{E}_{\|}\cdot\frac{\partial f_{i,e}}{\partial v_{\|}}=0.$$

The electron and ion distribution functions are reduced to a 3D phase space function $f_e(x, y, v_{\|e})$ and $f_i(x, y, v_{\|i})$, where $v_{\|}$ is the velocity variable along the magnetic field. The geometry of the system is given in Fig. 1, in which we have plotted the plasma box in the $x-y$ plane. We assume periodic boundary condition in the x-direction and zero boundary conditions in y-direction for electron and ion density and for the potential. That is, the y-direction is identified with the direction having non-periodic spatial variations. Thus we perform the numerical experiment in a rectangular domain with $0\leqslant x\leqslant L_x$ and $-L_y\leqslant y\leqslant L_y$ (only the $y>0$ part has been represented in Fig. 1).

Denoting by $-e$ and m_e (e and m_i) the electron (resp. ion) charge and mass we have

$$\frac{\partial f_e}{\partial t}+\left(v_{\|e}\cos\theta+\frac{E_y}{B}\sin\theta\right)\frac{\partial f_e}{\partial x} -\frac{E_x\sin\theta}{B}\frac{\partial f_e}{\partial y}-\frac{eE_x}{m_e}\cos\theta\frac{\partial f_e}{\partial v_{\|e}}=0. \tag{1}$$

A similar equation can be obtained for the ions,

$$\frac{\partial f_i}{\partial t}+\left(v_{\|i}\cos\theta+\frac{E_y\sin\theta}{B}\right)\frac{\partial f_i}{\partial x} -\frac{E_x\sin\theta}{B}\frac{\partial f_i}{\partial y}+\frac{eE_x}{m_i}\cos\theta\frac{\partial f_i}{\partial v_{\|i}}=0. \tag{2}$$

The electric fields are then given by

$$E_x=-\partial\phi/\partial x \tag{3}$$

and

$$E_y=-\partial\phi/\partial y, \tag{4}$$

where the electric potential φ obeys Poissons's equation,

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=+\frac{e}{\varepsilon_0}[n_e(x, y, t)-n_i(x, y, t)], \tag{5}$$

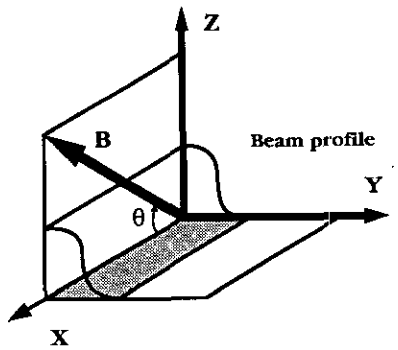

FIG. 1. A sketch of the two-dimensional simulation box x is the periodic dimension.

where n_e and n_i denote respectively the electron and ion densities given by

$$n_e(x, y, t) = \int f_e(x, y, v_{\|e})\, dv_{\|e} \tag{6}$$

$$n_i(x, y, t) = \int f_i(x, y, v_{\|i})\, dv_{\|i}. \tag{7}$$

The numerical integration of Eq. (1) is performed by using a splitting scheme in which we separate the integration in both directions successively according to the sequence of operators $(\hat{X}/2)(\hat{Y}/2)\,\hat{V}_{\|}(\hat{Y}/2)(\hat{X}/2)$ where the operators $\hat{X}$, $\hat{Y}$, and $\hat{V}_{\|}$ denote the shift in x, y, and $v_{\|}$ directions for both particle species, and over a full time step Δt. (The term $\frac{1}{2}$ denotes that the shift is effected over a half time step). Thus the fractional step method of Eq. (1), in the case of electrons for instance, involves five integration steps:

(A1) Between t_n and $t_{n+1/2}$ we shift the electron distribution function in x-space for a time $\Delta t/2$ and we obtain

$$f_e^*(x, y, v_{\|e}, t_{n+1/2}) = f_e\left[x - \left(v_{\|e}\cos\theta + \frac{E_y^{*n+1/4}\sin\theta}{B}\right)\frac{\Delta t}{2}, y, v_{\|e}, t_n\right]. \tag{8}$$

Note that this shift implies the computation of the y-component electric field $E_y^{*n+1/4}$ at time $t_{n+1/4} = (n+\frac{1}{4})\,\Delta t$. We have two possibilities to evaluate this electric field: first the Poisson's equation (this necessitates using a predictor-corrector method because we do not know the distribution function at time $t_{n+1/4}$ and it will be discussed in Section 2.3); on the other hand, we can use Maxwell Faraday's equation: the star in E_y^* formulation refers to this second choice (see Section 2.2 for the calculation of E_y^*).

(A2) We compute $E_x^{*n+1/4} = E_x(x, y, t_{n+1/4})$ and we shift the distribution f_e now along the x-axis for a time $\Delta t/2$ which yields

$$f_e^{**}(x, y, v_{\|e}, t_{n+1/2}) = f_e^*\left(x, y + E_x^{*n+1/4}\frac{\sin\theta}{B}\frac{\Delta t}{2}, v_{\|e}, t_{n+1/2}\right). \tag{9}$$

(A3) We compute the electric field components at time $t_{n+1/2}$ by solving Poisson's equation (this time the distribution function is known at time t_{n+1}) and we obtain $E_y^{n+1/2}$ and $E_x^{n+1/2}$; then we shift the distribution function in $v_{\|e}$ space for a full time step Δt according to

$$f_e^{3*}(x, y, v_{\|e}, t_{n+1/2}) = f_e^{**}\left(x, y, \left(v_{\|e} + \frac{e}{m_e}E_x^{n+1/2}\cos\theta\right)\Delta t, t_{n+1/2}\right). \tag{10}$$

(A4) Between $t_{n+1/2}$ and t_{n+1} we shift again the distribution function in y-space for half a time step after computing the x-component $E_x^{*n+3/4}$ at time $t_{n+3/4} = (n+\frac{3}{4})\,\Delta t$ and we obtain

$$f_e^{4*}(x, y, v_{\|e}, t_{n+1}) = f_e^{3*}\left(x, y + E_x^{*n+3/4}\frac{\sin\theta}{B}\frac{\Delta t}{2}, v_{\|e}, t_{n+1/2}\right). \tag{11}$$

(A5) To end up we shift f_e in the x-direction for a time $\Delta t/2$ after computing the y-component of the electric field $E_y^{*n+3/4}$ at time $t_{n+3/4}$ and we have

$$f_e(x, y, v_{\|e}, t_{n+1}) = f_e^{4*}\left[x - \left(v_{\|e}\cos\theta + E_y^{*n+3/4}\frac{\sin\theta}{B}\right) \times \Delta t, y, v_{\|e}, t_{n+1/2}\right]. \tag{12}$$

This sequence allows us to compute the electron distribution function f_e (or f_i in a similar way) at each grid points at time t_{n+1} from the known values of f_e (or f_i), and this scheme is correct in the second order in Δt. Thus we have reduced the integration of the Vlasov equation given by Eq. (1) to five successive interpolation problems. A very powerful method using cubic spline interpolation has been already used in the case of one-dimensional plasma (see Refs. [18, 19]) and it has been applied here.

2.2. *The Electric Field Computation*

In the previous integration steps from (A1) to (A5), the component field E_x^* and E_y^* are related to the integration of the Maswell–Faraday equation:

$$\varepsilon_0\frac{\partial \mathbf{E}}{\partial t} = -\mathbf{J} \tag{13}$$

since the magnetic field is uniform.

Equation (13) has been solved between $t_{n-1/4}$ and $t_{n+1/4}$ using a time centered scheme, provided that the electric field components are known at time t_n (which amounts to integrating Poisson's equation at time t_n). The current density $\mathbf{J}$ used in Eq. (13) can be written in the form

$$\mathbf{J} = \mathbf{J}_{\|} + \mathbf{J}_{\perp},$$

where

$$\mathbf{J}_{\perp} = -e\frac{\mathbf{E}\times\mathbf{B}}{B^2}(n_e(x, y, t) - n_i(x, y, t)) \tag{14}$$

the predicted value obtained by the linear analysis of the Rayleigh stability equation. Furthermore, the code provides an excellent spatial resolution which allows a detailed examination of the mutual interaction of vortices' rotating arms around the central structure. Then the extension of the model to an electrostatic drift-linetic plasma including electrons and ions motion in a 3D phase space has been effected in order to investigate the influence of a uniform tilted magnetic field on the formation of the vortex structures. A shift of the unstable wave number to the first harmonic has been observed which leads, in the case of $\theta = 80°$ to the appearance of two vortices in both electron and ion densities. Then for an angle $\theta = 75°$ a complete stabilization of the KH instability has been observed.

In Section 5, we have presented the results of a simulation for the ion temperature gradient instability. The very low noise level of the code allows us to study accurately an instability with a growth rate $\mathrm{Im}\,\omega/\omega_{pi} \sim 10^{-4}$. Further application of this code to ion temperature gradient instability is underway, especially by including in the equations the gyro-averaged effect of the ion Larmor radius, as indicated in [15], for instance.

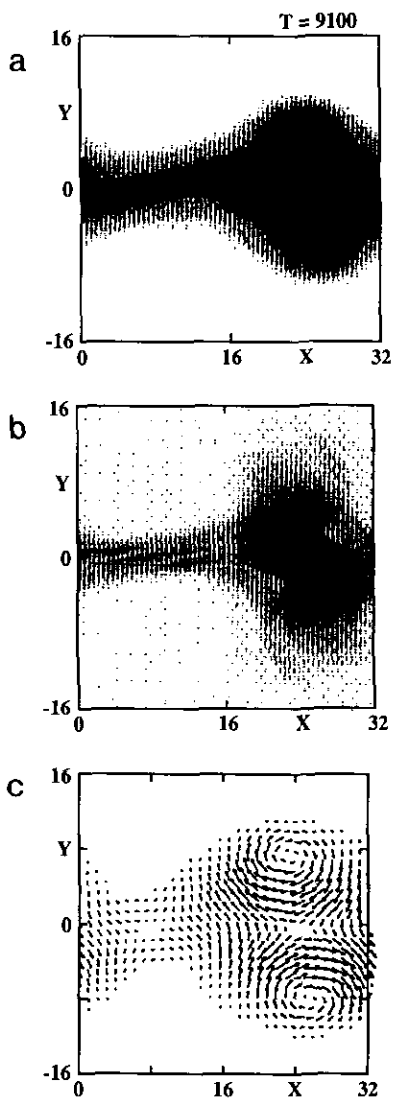

FIG. 18. (a) Ion density distribution in the $x-y$ plane showing the occurrence of a vortex structure at time $t\omega_{pi} = 9100$; (b) the corresponding kinetic temperature in the $x-y$ plane at time $t\omega_{pi} = 9100$; (c) The corresponding current density $n_i \mathbf{E} \times \mathbf{B}/B^2$ at time $t\omega_{pi} = 9100$.

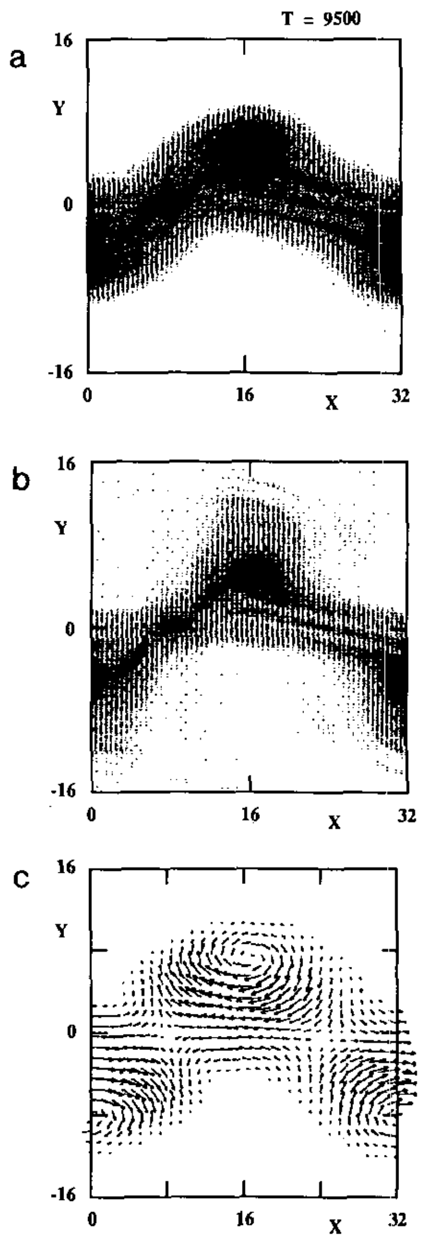

FIG. 19. (a) Ion density distribution at saturation at time $t\omega_{pi} = 9500$; (b) the corresponding kinetic temperature at time $t\omega_{pi} = 9500$; (c) the corresponding current flow $n_i \mathbf{E} \times \mathbf{B}/B^2$.

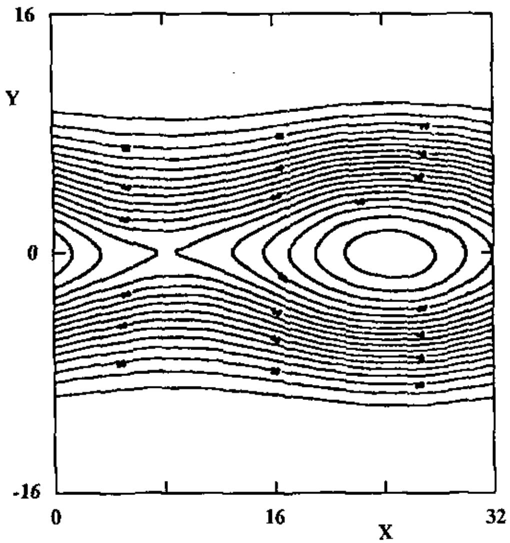

FIG. 20. Contour plot of the electron density at saturation.

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